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A DUOPOLY MODEL OF PRICING FOR INVENTORY
LIQUIDATION

by

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A Duopoly Model of Pricing for Inventory Liquidation

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We consider the problem of pricing to liquidate inventory in a duopoly. The problem is modelled as a multi-player game with complete information. The unique Nash equilibrium for the game, which is also sequential, involves mixed strategies for the sellers. The equilibrium extends in a limited fashion to multi-period and many seller situations.

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1 Introduction

Consider the problem faced by two firms seeking to liquidate their supplies of an identical product. At most one firm has supply sufficient to satisfy the entire market, and the two may not act in concert in setting price. To simplify matters, we assume a set of identical potential buyers each with a reservation price of 1 and a demand for exactly one unit of the product. Buyers, upon receipt of the price offered by the two sellers, are assumed to act independently.

The problem we discuss, while abstracted to allow analytical tractability, captures a large class of real world problems. The competition between opposing car dealerships at the end of a model year, might entail the sort of price competition described here. Alternatively, the model might aptly describe the situation faced by retailers liquidating their season-end stocks of a fashion item. Our analysis is directed toward the short-run problem of pricing to liquidate inventory rather than the long-run problem of price determination in a market. The literature on price dispersion is, however, relevant to our study.

The recent history of the economic literature on price determination under duopolistic competition is presented in some detail in Shubik (1984). A closely related line of research is directed to explaining how price dispersion can arise in a market. Casual observation reveals that identical products may sell for many different prices at a given point in time. Starting with the seminal work of Stigler (1961) two general approaches have been followed. One explains price dispersion as a by product of imperfect information on the part of consumers. This may result in transient price dispersion, caused during price adjustment (Diamond (1970)), or, in the presence of costly information, may result in persistent price dispersion (as in Salop and Stiglitz (1977) and Salop (1977)). Another approach, similar to the literature on monopolistic competition, makes use of heterogeneity among either the sellers or buyers (for example, the geographical distribution of sellers may be used to segment a market into local monopolies, as in Shilony (1977) and Butters (1977)). Finally, heterogeneity among consumer preferences may be used to obtain a temporal dispersion of prices as sellers try to discriminate among consumer types (Sobel 1984).

In our model, buyers and sellers are homogeneous, and buyers are perfectly informed about the prices offered by the sellers. We approach the problem as a non-cooperative game.

Our characterization of equilibrium behavior stems from a game theoretic analysis of the situation rather than from either incomplete information or from exogenous heterogeneities which serve to restrict the interactions between the sellers. We are motivated by the recent success of game-theoretic techniques in the analysis of the bargaining problem and the apparent similarity between the inventory liquidation problem and the classical bargaining problem. At issue in the liquidation problem is how to divide the value of the inventory between the sellers and the buyers. Indeed, were there exactly one seller, one buyer, and one unit of product, our problem would be a version of the classical bargaining problem. The main difficulty with the game theoretic approach is that it need not identify unique equilibrium conditions. There may exist a large number of equilibria. This problem arises in the bargaining problem: Nash equilibrium prices comprise a continuum between 0 and 1. Rubinstein (1982) provided a fundamental insight into the structure of this problem (and its extension to multiple periods) by restricting attention to sub-game perfect equilibria (see Selten (1965). Roughly, sub-game perfect equilibria require that the players eschew strategies which are not credible (i.e., players may not threaten to take actions which are not in their own self interest). For the bargaining problem and its multiperiod extension, Rubinstein obtains a unique sub-game perfect equilibrium.

An appropriate equilibrium concept is that of sequential equilibrium. This is a refinement of Nash equilibrium, in that optimality is required not only along the equilibrium path but also off the equilibrium path. See Kreps and Wilson (1982) for details. We obtain a unique Nash equilibrium. This equilibrium is also sequential. In this equilibrium, sellers randomize on an interval between a price of 1 and a lower threshold. The mixed strategy of the seller with a higher supply stochastically dominates the mixed strategy of the other seller. This randomization drives the price dispersion for the market. Thus, our equilibrium is characterized by randomly chosen prices, rather than a set of distinct but deterministic prices (the common result of models with producer or consumer heterogeneity). In this respect our results are similar to Shilony's, although he invokes locational heterogeneity of producers. In Section 2 of this paper, we present the details of the model and the equilibrium strategies. The results proceed by way of a sequence of lemmas characterizing the nature of equilibrium behaviour. Section 3 explores briefly the robustness of our proposed equilibrium when there are either many periods in which to liquidate the inventory or more

than two sellers. We will see that there are versions of the equilibrium which hold in each case, however the equilibrium is no longer unique. Conclusions are presented in Section 4.

2 The Basic Model

There are m risk-neutral buyers of an indivisible commodity, and two (also risk neutral) sellers, called 1 and 2, endowed with n_1 and n_2 units of the product, respectively. The value to a buyer of a unit of the product is 1. Each buyer demands exactly one unit. The unit cost to the sellers 0. If the buyer obtains a unit of the good at price p , then his utility is $(1 - p)$, if he does not trade his utility is 0. If a seller sells s units at price p , then his utility is sp . The sellers are assumed to set prices independently and simultaneously, and buyers make their purchase decision independently. We view the price setting problem as a non-cooperative game in which sellers and buyers must divide the value of the units sold among themselves. The "actions" available to the players are price offers (for sellers) and accept/reject decisions (for buyers).

We start with the simplest possible model. All trade must take place in a single period there are exactly two sellers and there are many buyers. In section 3 we will explore the impact of changing our assumptions about the timing of sales and the number of sellers.

The actions available to the sellers are price offers which are assumed to be made simultaneously and independently. Buyers may accept or reject the offered prices. If there is excess demand then the available supply is allocated randomly over the buyers. If there is excess supply at a given price then demand is allocated randomly between the sellers.

The sellers make simultaneous price offers p_1 and p_2 respectively. Without further loss of generality, we can restrict the sellers price offers to $[0, 1]$. The buyers' actions are either accept or reject for each price. All this is common knowledge among the buyers and the sellers. This defines a game with complete information. We show that this game has a unique Nash equilibrium in which the sellers use randomized strategies. This equilibrium is also sequential.

The buyers' optimal strategy is straightforward. They accept the lower price and buy from the higher price seller only after the low price seller's supply is exhausted. In addition, they reject any price greater than 1. The interesting question revolves around the

equilibrium price offers of the sellers. The strategy for seller i , $i = 1, 2$, is denoted by $F_i(p)$, a probability distribution function on $[0, 1]$. We allow the possibility that F_i assigns all its probability mass to a point. Two trivial cases may be dispensed with immediately: if $n_1 + n_2 \leq m$ then the unique equilibrium is for both sellers to quote a price of 1 and for the buyers to accept the price; if $\min(n_1, n_2) \geq m$ then both sellers offer a price of 0 and all buyers accept.

For the remainder of this section we will assume that $n_1 > 0$, $n_2 > 0$, $\min(n_1, n_2) < m$ and $n_1 + n_2 > m$. Let $\pi^i(p_1, p_2)$ denote the payoff to seller i when seller 1 offers p_1 and seller 2 offers p_2 and the buyers play their best response, that is, they buy from the cheapest seller first.

$$\pi^1(p_1, p_2) = \begin{cases} n_1 p_1, & \text{if } p_1 < p_2 \\ \max\{(m - n_2)p_1, 0\}, & \text{if } p_1 > p_2 \\ m\left(\frac{n_1}{n_1 + n_2}\right)p_1, & \text{if } p_1 = p_2 \end{cases}$$

$$\pi^2(p_1, p_2) = \begin{cases} n_2 p_2, & \text{if } p_2 < p_1 \\ \max\{(m - n_1)p_2, 0\}, & \text{if } p_2 > p_1 \\ m\left(\frac{n_2}{n_1 + n_2}\right)p_2, & \text{if } p_1 = p_2 \end{cases}$$

Denote by $\Gamma^1(p_1, F_2)$ ($\Gamma^2(F_1, p_2$)) to payoff to player 1 (2) when he uses a pure strategy p_1 (p_2) and his opponent uses the mixed strategy F_2 (F_1).

$$\Gamma^1(p_1, F_2) = \int_0^1 \pi^1(p_1, p_2) dF_2(p_2)$$

$$\Gamma^2(F_1, p_2) = \int_0^1 \pi^2(p_1, p_2) dF_1(p_1).$$

The following characterizes any Nash equilibrium of this game.

PROPOSITION 1: Suppose that $n_1 > 0$, $n_2 > 0$, $n_1 + n_2 > m$, $\min(n_1, n_2) < m$. Let (F_1, F_2) be the sellers strategies in a Nash equilibrium in this game. Then,

- a) the support of F_i , $i = 1, 2$, is $[\underline{a}, 1]$, $\underline{a} > 0$;
- b) F_i , $i = 1, 2$ is atomless on $[\underline{a}, 1]$;
- c) if $z_i(1)$ is the probability mass assigned by F_i at 1 then $z_1(1)z_2(1) = 0$.

Proposition 1 implies:

COROLLARY 1: *There does not exist an equilibrium in which either seller employs a pure strategy.*

Proposition 1 is proved through Lemmas 1-5 below.

LEMMA 1: Suppose that (F_1, F_2) are the sellers' strategies in an equilibrium. Let $\underline{a}^i = \inf\{x \in [0, 1] | F_i(x) > 0\}$, $i = 1, 2$. Then,

a) $\underline{a}^1 = \underline{a}^2 \equiv \underline{a}$;

b) $\underline{a} > 0$.

PROOF: a) Suppose $\underline{a}^1 > \underline{a}^2$. Let ϵ_1 be such that $\underline{a}^1 - \underline{a}^2 > \epsilon_1 > 0$. Then seller 2 can improve his payoff by modifying his mixed strategy so that he charges a price of $\underline{a}^2 + \epsilon_1$ whenever his equilibrium mixed strategy requires him to quote a price in the interval $[\underline{a}^2, \underline{a}^2 + \epsilon_1)$, because

$$n_2(\underline{a}^2 + \epsilon_1) > n_2(\underline{a}^2 + x), \quad \forall x \in [\underline{a}^2, \underline{a}^2 + \epsilon_1).$$

Thus $\underline{a}^1 \leq \underline{a}^2$. A symmetric argument establishes that $\underline{a}^2 \leq \underline{a}^1$. Therefore, we must have, $\underline{a}^1 = \underline{a}^2 \equiv \underline{a}$.

b) Suppose $\underline{a} = 0$ and that $n_1 \geq n_2$. Then, since $n_2 < m$, seller 1 is better off quoting a price of 1 rather than a price in the interval $[0, \epsilon)$, for small enough, positive ϵ . Hence $\underline{a} > 0$. ■

LEMMA 2: Suppose that (F_1, F_2) are the sellers' strategies in an equilibrium. Let $\bar{a}^i = \sup\{x \in [0, 1] | F_i(x) < 1\}$, $i = 1, 2$. Then,

a) $\bar{a}^1 = \bar{a}^2 \equiv \bar{a}$,

b) $\bar{a} = 1$.

PROOF: a) Suppose that $\bar{a}^2 > \bar{a}^1$ and $n_1 \geq m$. Then $\Gamma^2(F_1, p) > \Gamma^2(F_1, \hat{p}) = 0$, $\forall p \in (\bar{a}^1, \underline{a})$, $\forall \hat{p} > \bar{a}^1$. Thus, F_2 cannot be a best response.

On the other hand, suppose that $\bar{a}^2 > \bar{a}^1$ and $n_1 < m$. Then $\Gamma^2(F_1, p) = (m - n_1)p < (m - n_1) = \Gamma^2(F_1, 1)$, $\forall p \in (\bar{a}^1, 1)$. Therefore, $\bar{a}^2 = 1$, $F_2(p) - F_2(\bar{a}^1) = 0$, $\forall p \in (\bar{a}^1, 1)$ and, from the definition of \bar{a}^2 , $F_2(1) - F_2(1^-) > 0$. Let $z_2(1) \equiv F_2(1) - F_2(1^-)$.

Next, suppose that, $z_2(\bar{a}^1)$, the probability mass assigned by F_2 at \bar{a}^1 , is zero. Then, from the fact that $F_2(\bar{a}^1) - F_2(\bar{a}^1 - \epsilon) \rightarrow 0$, as $\epsilon \rightarrow 0$, it follows that $\Gamma^1(p, F_2) < \Gamma^1(1 -$

$\epsilon_0, F_2)$, $\forall p \in (\bar{a}^1 - \epsilon_0, \bar{a}^1]$, for small enough $\epsilon_0 > 0$. But this contradicts the fact that F_1 is a best response. Therefore, $z_2(\bar{a}^1) > 0$. This in turn implies that $\Gamma^1(\bar{a}^1, F_2) < \Gamma^1(\bar{a}^1 - \epsilon, F_2)$, for all small enough $\epsilon > 0$. Therefore, $z_1(\bar{a}^1)$, the probability mass assigned by F_1 at \bar{a}^1 must be zero. But then, $\Gamma^2(F_1, 1) = (m - n_1) > (m - n_1)\bar{a}^1 = \Gamma^2(F_1, \bar{a}^1)$. Therefore, if $\bar{a}^2 > \bar{a}^1$, then if $z_2(\bar{a}^1) > 0$, F_2 cannot be a best response to F_1 , whereas if $z_2(\bar{a}^1) = 0$, then F_1 cannot be a best response to F_2 . Hence $\bar{a}^2 \leq \bar{a}^1$. Similarly, $\bar{a}^2 \geq \bar{a}^1$ and thus, in any equilibrium, we must have $\bar{a}^1 = \bar{a}^2 \equiv \bar{a}$.

b) Next, suppose that $\bar{a} < 1$. Then, by an argument similar to the one in the preceding paragraph, we can show that either $z_1(\bar{a})$ or $z_2(\bar{a})$ is equal to zero. Suppose that $z_2(\bar{a}) = 0$. But then, for small enough, positive ϵ , $\Gamma^1(p, F_2) < \Gamma^1(1, F_2)$, $\forall p \in (\bar{a} - \epsilon, \bar{a}]$. This contradicts our assumption that F_1 is a best response. ■

LEMMA 3: *In any equilibrium, the sellers' strategies cannot have a mass point at a price less than 1, i.e., there does not exist $x < 1$, such that $F_i(x) - F_i(x^-) > 0$, $i = 1, 2$.*

PROOF: Suppose that for an equilibrium pair of mixed strategies F_1 and F_2 , there exists $x < 1$ such that $F_1(x) - F_1(x^-) = z_1(x) > 0$. Suppose that for all $\epsilon > 0$, $F_2(x + \epsilon) - F_2(x) > 0$. Then seller 2 would do better to modify his mixed strategy to F'_2 , where F'_2 is obtained from F_2 by transferring the probability mass in the interval $(x, x + \epsilon)$ to $x - \epsilon$. This follows from:

$$\Gamma^2(F_1, y) \leq (1 - F_1(x))n_2(x + \epsilon) + F_1(x)(x + \epsilon) \max(m - n_1, 0), \quad \forall y \in (x, x + \epsilon),$$

$$\Gamma^2(F_1, x - \epsilon) \geq (1 - F_1(x^-))n_2(x - \epsilon) + F_1(x^-)(x - \epsilon) \max(m - n_1, 0).$$

Therefore,

$$\Gamma^2(F_1, x - \epsilon) - \Gamma^2(F_1, y) \geq z_1(x)x \min(n_1 + n_2 - m, n_2) - o(\epsilon), \quad \forall y \in (x, x + \epsilon),$$

which is positive for small enough ϵ . But this contradicts our hypothesis that F_2 is a best response. Hence there exists $\epsilon_0 > 0$ such that $F_2(x + \epsilon_0) = F_2(x)$. By a similar argument, $F_2(x) = F_2(x^-)$.

From Lemma 2 we know that $F_2(x) < 1$. But then, seller 1 could do better by transferring the probability mass at x to $x + \epsilon_0/2$. Hence, F_1 cannot be a best response to F_2 . ■

LEMMA 4: *In any equilibrium, at most one seller's strategy can have a mass point at a price of one.*

PROOF: If both sellers use strategies which have an atom at one, then any seller can do better by shifting the atom to a price slightly less than one. ■

LEMMA 5: [No "gaps" in equilibrium strategies.]

In any equilibrium, if $F_i(x) \in (0, 1)$, $i = 1, 2$, then $\forall y > x$, $F_i(y) > F_i(x)$.

PROOF: Suppose that $F_1(x) \in (0, 1)$ and that there exists $y \in (x, 1)$ such that $F_1(y) = F_1(x)$. Then, $F_2(y) = F_2(x)$; otherwise if seller 2 modifies his strategy and shifts the probability mass in the interval (x, y) to y , his payoff improves.

Without loss of generality, we may assume that $F_1(x) - F_1(x - \epsilon) > 0$, $\forall \epsilon > 0$. By Lemma 2 we know that $F_2(x) < 1$, and by Lemma 3 that F_1 and F_2 are atomless at prices below one. Then, for small enough $\epsilon_0 > 0$, if seller 1 transfers the probability mass assigned by F_1 to the interval $[x - \epsilon_0, x]$ to y , his payoff improves. This contradicts the hypothesis that F_1 is a best response. ■

The proof of Proposition 1 is immediate.

Proposition 1 states the necessary conditions for any equilibrium in the game. We will make use of this structure, and show that there exists a unique equilibrium. (The equilibrium is unique in the sense that the players' strategies in any other equilibrium differ only on a set of measure zero).

Suppose that $n_1 + n_2 > m$, $n_1 \geq n_2 > 0$ and $m > n_2$. Let $((z_1(1), \hat{F}_1), (z_2(1), \hat{F}_2))$ be a pair of equilibrium strategies. Seller i quotes a price of 1 with probability $z_i(1)$, and with probability $1 - z_i(1)$ draws a price from the atomless distribution \hat{F}_i , which has support $[\underline{a}, 1]$. In addition, $z_1(1)z_2(1) = 0$. From Proposition 1 we know that any equilibrium must have this structure.

Suppose that $z_2(1) > 0$. Then $z_1(1) = 0$. If $n_1 \geq m$, then clearly seller 2 does better by moving the probability mass at 1, $z_2(1)$, to any price less than 1. Therefore assume that $n_1 < m$. Since, in any equilibrium, seller 2 must be indifferent between offering a price of 1 and almost every price in the interval $[\underline{a}, 1]$, we have,

$$m - n_1 = n_2(\underline{a} + \epsilon)(1 - \hat{F}_1(\underline{a} + \epsilon)) + (m - n_1)(\underline{a} + \epsilon)\hat{F}_1(\underline{a} + \epsilon), \text{ for almost all } \epsilon \in (0, 1 - \underline{a}).$$

Letting $\epsilon \searrow 0$, we get

$$m - n_1 = n_2 \underline{a} \text{ or } \underline{a} = \frac{m - n_1}{n_2}$$

Similarly, since seller 1 is indifferent between offering almost any price close to 1, and almost any price close to \underline{a} , we have

$$(m - n_2)(1 - z_2(1)) + n_1 z_2(1) = n_1 \underline{a} = n_1 \frac{(m - n_1)}{n_2},$$

$$\Rightarrow z_2(1) = \frac{n_2 - n_1}{n_2} \leq 0,$$

as $n_1 \geq n_2$. This contradicts our assumption that $z_2(1) > 0$. Therefore we must have $z_2(1) = 0$. A similar argument shows that

$$\underline{a} = \frac{m - n_2}{n_1}. \quad (1)$$

and,

$$z_1(1) = \frac{\min(m, n_1) - n_2}{n_1}. \quad (2)$$

Thus in any equilibrium, \underline{a} , $z_2(1)$, and $z_1(1)$ must have the values specified above. Next, we show that \hat{F}_1 , \hat{F}_2 , must be unique except on a set of measure zero. The next equation states that player 1 must be indifferent between offering a price of 1 and any other price in $[\underline{a}, 1]$.

$$m - n_2 = (1 - \hat{F}_2(x))n_1 x + \hat{F}_2(x)(m - n_2)x, \quad \forall x \in [\underline{a}, 1].$$

(The above equation need hold only for almost all x , for an equilibrium. Thus the equilibrium we obtain, is unique except on a set of measure zero).

Therefore,

$$\hat{F}_2(x) = \frac{n_1}{n_1 + n_2 - m} \left(1 - \frac{\underline{a}}{x}\right), \quad \forall x \in [\underline{a}, 1]. \quad (3)$$

Similarly

$$z_1(1)n_2 + (1 - z_1(1)) \min(m - n_1, 0) =$$

$$(z_1(1) + (1 - z_1(1))(1 - \hat{F}_1(x)))n_2 x + (1 - z_1(1))\hat{F}_1(x) \min(m - n_1, 0)x, \quad \forall x \in [\underline{a}, 1].$$

Therefore

$$\hat{F}_1(x) = \frac{n_1}{n_1 + n_2 - m} \left(1 - \frac{\underline{a}}{x}\right), \quad \forall x \in [\underline{a}, 1]. \quad (4)$$

Hence, we have proved:

PROPOSITION 2: Suppose $n_1 + n_2 > m$, $n_1, n_2 > 0$, $n_1 \geq n_2$, $m > n_2$. Then, in the unique Nash equilibrium of this game:

- a) The buyers accept any price in the interval $[0, 1]$; they prefer to buy from the seller who offers the lower price, and if unable to do so, they buy from the other seller.
- b) Seller 1 offers a price of 1 with probability $z_1(1)$, and with probability $(1 - z_1(1))$ draws a price from the atomless distribution \hat{F} ; seller 2 offers a price from the same distribution \hat{F} , where

$$z_1(1) = \frac{\min(m, n_1) - n_2}{n_1},$$

$$\hat{F}(x) = \frac{n_1}{n_1 + n_2 - m} \left(1 - \frac{a}{x}\right), \quad \forall x \in [a, 1],$$

$$a = \frac{m - n_2}{n_1}.$$

This equilibrium is also sequential.

The nature of our equilibrium reflects the intuition one obtains from considering the obvious cooperative arrangement that the sellers might make: one seller agrees to sell immediately at a price low enough to induce the buyers to accept right away, and the other seller sells to the remaining buyers at the monopoly price. The difficulty lies in how to enforce such an arrangement absent the ability to collude on price. Any price which results in profitable early sales encourages undercutting. Indeed, were the sellers restricted to pure strategies, it is clear that any price above the lower threshold invites undercutting; however, the best response to a price equal to the lower threshold is a price of 1. Our non-cooperative analysis results in unique randomized equilibrium strategies. The sellers randomize continuously between a price of 1 and some lower threshold. The supplier with the larger supply also has a positive probability of charging exactly 1. Buyers buy immediately starting with the seller with the lower price. By randomizing, players are able to hide their intentions, thereby eliminating the motivation for undercutting which would be present were the players restricted to pure strategies.

3 Many Sellers and Multiple Periods

Our purpose here is not to propose our equilibrium as the general solution for a market

game (which it is not), but to illuminate it's robustness with respect to small deviations from the pure duopoly, single period model.

3.1 The Many Sellers Model

Suppose now that there are k sellers with supplies $n_1 > n_2 \geq n_3 \geq \dots \geq n_k$ and the game proceeds as described in section 2. We will see that our equilibrium extends to the multiseller case in which there is one large sellers and a collection of small sellers.

In the equilibrium of Proposition 3 below, there are k sellers. Sellers 3 through k sell at a price \underline{a} , and sellers 1 and 2 use the equilibrium strategies of section 2, randomizing between \underline{a} and 1. Our assumptions amount to requiring that the supplies of each of sellers 2 through k are quite small in comparison to the supplies of seller 1. There exist $k - 2$ other equilibria, which are similar in structure. In these equilibria, sellers 1 and j , $j \geq 2$, use the equilibrium strategies of section 2, randomizing between \underline{a} and 1, while the other sellers charge a price of \underline{a} .

PROPOSITION 3: Suppose that $n_1 > n_2 \geq \dots \geq n_k$ satisfy $\sum_{i=2}^k n_i < m$, and for each $j = 3, \dots, k$, $\sum_{i \neq 2, j} n_i > m$. Define $N_{-i} = \sum_{l \neq i} n_l$, $\underline{a} = (m - N_{-1})/n_1$, $z_1(1) = (\min(m, N_{-2}) - N_{-1})/n_1$, and $\hat{F}(x) = (1 - \underline{a}/x)n_1/(N - m)$. Then a sequential equilibrium for the game is given by the following:

- a) The buyers accept any price in the interval $[0, 1]$ starting with the lowest price seller.
- b) Sellers 3, ..., k offer price \underline{a} , seller 1 offers a price of 1 with probability $z_1(1)$ and chooses a price randomly according to $\hat{F}(x)$ with probability $1 - z_1(1)$, seller 2 chooses a price randomly according to $\hat{F}(x)$.

PROOF: As in Section 2, the buyers strategy is a best response. Proposition 2 implies that given the choices of players 3, ..., k , players 1 and 2's choices are optimal. We will show that player j ($j = 3, \dots, k$) will not wish to deviate. It is also clear that a price of \underline{a} dominates any smaller price for seller j . We need only show that player j cannot improve his expected payoff by offering a price greater than \underline{a} . Let $\Gamma^j(x)$ denote player j 's expected payoff when he offers a price of x and the other players play their equilibrium strategies. Then, under the hypotheses of the proposition, player j sells a non-zero quantity if and

only if player 1 offers a price higher than x . The amount he sells in this event is n_j . Thus,

$$\Gamma^j(x) = (1 - F_1(x))n_j x,$$

or

$$\Gamma^j(x) = (1 - (1 - z_1(1))\hat{F}(x))n_j x$$

Substituting from the definitions and from our assumptions on n_1 yields,

$$\Gamma^j(x) = n_3 \underline{a}, \quad \forall x \in [\underline{a}, 1].$$

■

In the equilibrium of Proposition 3 the largest seller and one of the other sellers play the duopoly pricing game ignoring the other smaller sellers. The equilibrium price for the smaller sellers is a by product of this competition.

3.2 The Multiple Period Model

Let $t = 0, 1, 2, \dots$ denote time periods. At time $t = 0$ the game starts as given in the preceding section. Buyers who accept a price offer in period t , exit the game upon receiving one unit of the good at that price. Buyers who reject the price offers in period t have another opportunity in period $t + 1$ to purchase the product. Sellers make a new price offer for their remaining units in each period. Let n_i^t denote the inventory remaining with seller i at the beginning of period t and let m^t denote the number of buyers who have yet to purchase a unit at the beginning of period t . We assume a common discount factor of δ per period for both buyers and sellers.

Once again we have trivial unique equilibria when $n_1^0 + n_2^0 \leq m^0$ or if $\min(n_1^0, n_2^0) \geq m$. In the former case the equilibrium price is 1 and in the latter 0. Henceforth, assume $n_1^0 + n_2^0 > m^0$, $\min(n_1^0, n_2^0) < m$, and $n_1^0 > n_2^0$. In a fashion similar to lemma 1 we have.

LEMMA 7: When $n_1^0 + n_2^0 > m$, $\min(n_1^0, n_2^0) < m$ then there is no Nash equilibrium involving pure strategies only.

Let $\bar{s}^t = (n_1^t, n_2^t, m^t)$, and

$$z_i(1; \bar{s}^t) = \max\{0, \frac{\min(m^t, n_i^t) - n_j^t}{n_i^t}\}, \quad (5)$$

$$\hat{F}(x; \tilde{s}^t) = \frac{\max(n_1^t, n_2^t)}{n_1^t + n_2^t - m^t} \left(1 - \frac{a^t}{x}\right), \quad \forall x \in [\underline{a}^t, 1], \quad (6)$$

where,

$$\underline{a}^t(\tilde{s}^t) = \frac{m^t - \min(n_1^t, n_2^t)}{\max(n_1^t, n_2^t)}. \quad (7)$$

The equilibrium outcome in the one period model remains an Nash equilibrium outcome in the multiple period game, when $\max(n_1^0, n_2^0) < m^0$. However, this equilibrium is not sequential because if an off-the-equilibrium path subgame starting at $n_i^t < m^t < n_j^t$ is reached, and seller j offers the lower price, then a single buyer is better off deviating, rejecting both offers and obtaining a price of 0 in the next period. Such a subgame may be reached if there are deviations by more than one player. Similarly, the equilibrium given below breaks down if $\max(n_1^0, n_2^0) \geq m^0 > \min(n_1^0, n_2^0)$.

PROPOSITION 4: Assume that $n_1^0 + n_2^0 > m^0$, $n_1^0, n_2^0 > 0$, $\max(n_1^0, n_2^0) < m^0$. Then, the following strategies comprise a Nash equilibrium in the infinite period setting:

Sellers Strategy: At any stage t , if $n_1^t > 0$, $n_2^t > 0$, then seller i offers a price of 1 with probability $z_i(1; \tilde{s}^t)$, and with probability $(1 - z_i(1; \tilde{s}^t))$ draws a price from the atomless distribution $\hat{F}(\cdot; \tilde{s}^t)$. If $n_i^t = 0$ and $m^t > 0$, then the seller j offers a price of 1.

Buyers Strategy: At any stage t , if $n_1^t > 0$, $n_2^t > 0$, accept the lower offer, if it is not greater than 1; if unable to buy from the seller who offers the lower price, then accept the other offer provided it is not greater than 1. If $n_i^t = 0$ then accept a price of 1 from seller j .

PROOF: Consider a buyer who is still left in the game in period t . If $n_i^t = 0$, then $n_j^t > 0$ and seller j will offer a price of 1 in every period τ , $\tau \geq t$. Hence the buyer cannot do better by rejecting the offer in period t . If $n_1^t > 0$, $n_2^t > 0$, then his strategy is optimal, because if the other buyers accept the sellers' equilibrium offers, and he rejects then from next period on there will only be one seller who will always offer a price of 1.

Consider a seller, say seller 1. If $n_2^t = 0$ and $m^t > 0$, then given the buyers strategy, he cannot do better than offer a price of 1. If $n_1^t > 0$, $n_2^t > 0$, $m^t > 0$, then there are two types of deviations by seller 1. An offer $p < \underline{a}^t(\tilde{s}^t)$, yields a lower payoff than offering $\underline{a}^t(\tilde{s}^t)$. Hence this deviation is not profitable. On the other hand, an offer $p > 1$ will be rejected by the buyers, n_2^t of whom will accept seller 2's equilibrium offer and exit the game, leaving seller 1 with an expected payoff of at most $(m^t - n_2^t)$ tomorrow, which is less than $(m^t - n_2^t)$ today, his payoff if he offers a price of 1 today. Given seller 2's and the buyers' strategies,

seller 1 is indifferent between offering any price between $\underline{a}'(\bar{s}')$ and 1. Hence his strategy is optimal. ■

4 Conclusion

We have established the existence of a unique Nash equilibrium which is also sequential for a somewhat stylized description of an inventory liquidation problem. Equilibrium play requires that the players make use of randomized strategies, and this results in price dispersion. The dispersion arises as players attempt to hide their intentions so as to avoid price undercutting. This notion of dispersion is different from the commonly obtained price dispersion result wherein sellers propose differing deterministic prices which take advantage of market segmentation.

We have established a limited robustness for our proposed equilibrium behavior. The equilibrium obtained in the two person, two period model remains an equilibrium when liquidation occurs over many periods, or if there are additional (small sellers). In these two cases however, we can no longer guarantee the uniqueness of the equilibrium.

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